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B.Sc. Part III 5th Paper

Infinite Products

Let U_n = any real function defined for all (+)ve integral values of n then the product $U_1 . U_2 . U_3 . \ldots . U_n$ is represented by P_n.

$$\therefore P_n = U_1 . U_2 . U_3 \ldots U_n = \prod_{r=1}^{n} U_r$$

If $P = U_1 . U_2 . U_3 \ldots$ to ∞ then $P = \prod_{r=1}^{\infty} U_r$

* Convergence of infinite products

If $\lim_{n \to \infty} P_n$ = a finite limit $P (\neq 0)$ then the infinite product $\prod_{n=1}^{\infty} U_n$ converges to the limit P and it is denoted as

$$\boxed{\prod_{1}^{\infty} U_n = \lim_{n \to \infty} P_n = P}$$

Note that none of the factors of P_n is zero.

Q1 Prove that the product

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots \text{ converges to}$$

a definite limit different from zero.

Soln.

Let $a_1 = 1-\frac{1}{2^2}$, $a_2 = 1-\frac{1}{3^2}$,

$a_3 = 1-\frac{1}{4^2}$, $\ldots$, $a_{n-1} = 1-\frac{1}{n^2}$, $\ldots$

Let $P_{n-1} = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$

$$\Rightarrow P_{n-1} = a_1 a_2 a_3 \cdots a_{n-1} \quad \text{--- (1)}$$

Now $a_{n-1} = 1-\frac{1}{n^2} = \frac{n^2-1}{n^2} = \frac{(n+1)(n-1)}{n^2} = \frac{(n-1)(n+1)}{n^2}$

Put $n=2$	Put $n=3$	Put $n=4$	$\cdots$
$\Rightarrow a_1 = \frac{1.3.}{2^2}$	$a_2 = \frac{2.4}{3^2}$	$a_3 = \frac{3.5}{4^2}$	$\cdots$

$$\Rightarrow P_{n-1} = a_1 . a_2 . a_3 \cdots a_{n-1}$$

$$\Rightarrow P_{n-1} = \frac{1.3}{2^2} . \frac{2.4}{3^2} . \frac{3.5}{4^2} \cdots \frac{(n-1)(n+1)}{n^2}$$

$$\Rightarrow P_{n-1} = \left(\frac{1}{2}.\cancel{\frac{3}{2}}\right)\cancel{\left(\frac{2}{3}.\frac{4}{3}\right)}\cancel{\left(\frac{3}{4}.\frac{5}{4}\right)}\cdots\left(\cancel{\frac{n-1}{n}}.\frac{n+1}{n}\right)$$

$$\Rightarrow P_{n-1} = \frac{1}{2} . \frac{n+1}{n}$$

$$\Rightarrow P_{n-1} = \frac{1}{2}\cdot\left(\frac{n+1}{n}\right) = \frac{1}{2}\cdot\left(1+\frac{1}{n}\right)$$

Take $\underset{n\to\infty}{Lt}$ both sides, we get

$$\Rightarrow \underset{n\to\infty}{Lt} P_{n-1} = \frac{1}{2}\cdot \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right) = \frac{1}{2}\cdot(1+0) = \frac{1}{2}$$

$$\text{Now, } P_n = P_{n-1} \times \left[1-\frac{1}{(n+1)^2}\right] = P_{n-1} \times \frac{(n+1)^2-1}{(n+1)^2}$$

$$\Rightarrow P_n = P_{n-1} \times \frac{n(n+2)}{(n+1)^2} = P_{n-1} \times \frac{\frac{n(n+2)}{n^2}}{\frac{(n+1)^2}{n^2}}$$

$$\Rightarrow P_n = P_{n-1} \times \frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2}$$

Take $\underset{n\to\infty}{Lt}$ both sides, we get

$$\underset{n\to\infty}{Lt} P_n = \underset{n\to\infty}{Lt} P_{n-1} \times \underset{n\to\infty}{Lt}\left[\frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^2}\right]$$

$$\Rightarrow \underset{n\to\infty}{Lt} P_n = \frac{1}{2} \times \frac{1+0}{1^2} = \frac{1}{2}.$$

So, the given product converges to a finite limit $\frac{1}{2}$, different from zero.